XXX. *Geometrical Solutions of three celebrated Aſtronomical Problems, by the late Dr.* Henry Pemberton, *F. R. S. Communicated by* Matthew Raper, *Eſq; F. R. S.*

LEMMA.

Read June 4, 1772.

TO form a triangle with two given ſides, that the rectangle under the ſine of the angle contained by the two given ſides, and the tangent of the angle oppoſite to the leſſer of the given ſides, ſhall be the greateſt that can be.

Let [TAB. XII. Fig. 1.] the two given ſides be equal to A B and A C: round the center A, with the interval A C, deſcribe the circle C D E, and produce B A to E; take B F a mean proportional between B E and B C, and erect the perpendicular F G, and complete the triangle A G B.

Here the ſine of B A G is to the radius, as F G to A G; and the tangent of A B G to the radius, as F G to F B: therefore, the rectangle under the ſine of B A G and the tangent of A B G is to the ſquare of the

Fig. 1.

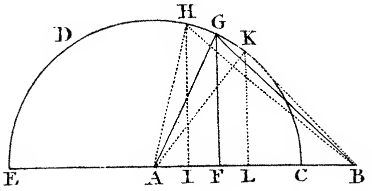
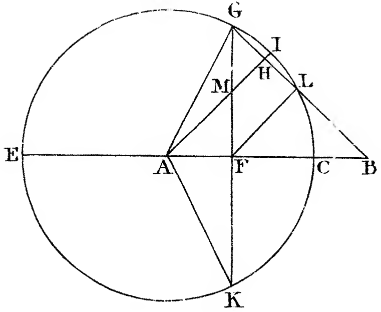


Fig. 2.



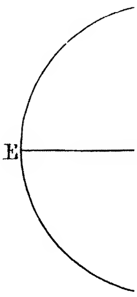


Fig. 4.

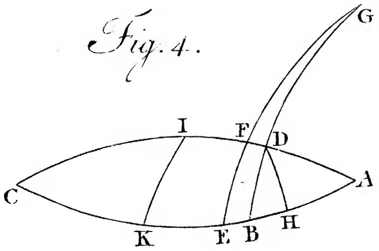


Fig. 5.

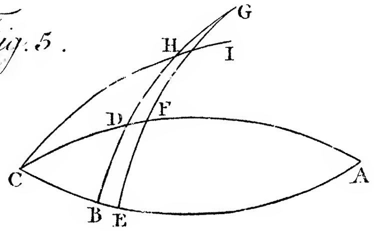
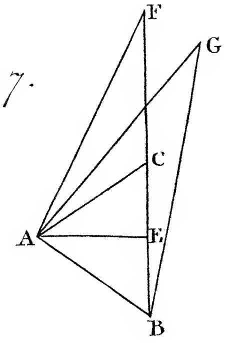


Fig. 6.

Fig. 7.



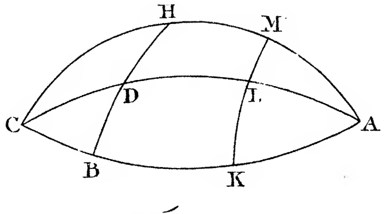


Fig. 8.

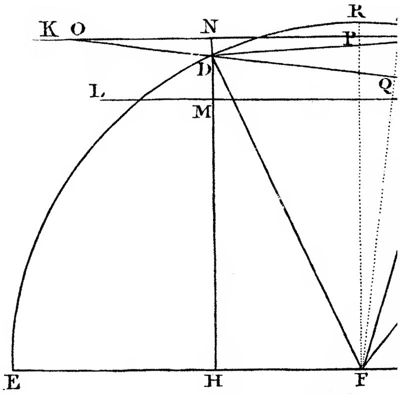
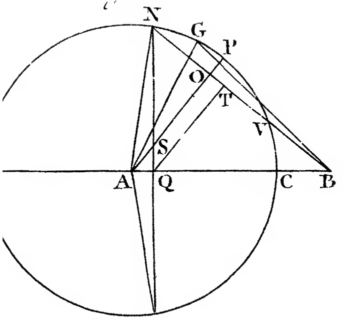


Fig. 9

Fig. 3.



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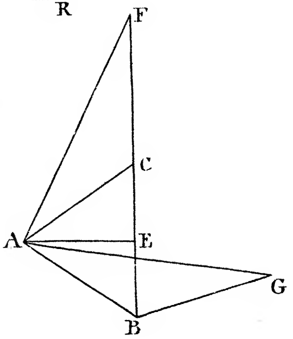

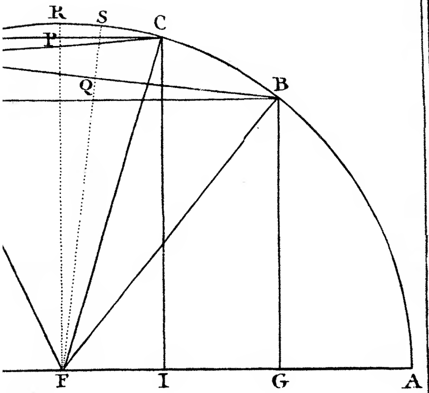

Fig. 9.

Basire sc.

the radius, as the ſquare of FG, or the rectangle EFC, to the rectangle under AG (or AC) and FB. But, EB being to BF as BF to BC, by converſion, EB is to EF as BF to FC, and alſo, by taking the difference of the antecedents and of the conſequents, EF is to twice AF as BF to FC; and twice AFB is equal to EFC.

Now, let the triangle BAH be formed, where the angle BAH is greater than BAG. Here, the perpendicular HI being drawn, the rectangle under the ſine of BAH and the tangent of ABH will be to the ſquare of the radius, as the rectangle EIC to the rectangle under AC, IB. But IF is to FB as 2AFI to 2AFB, or EFC; and 2AFI is greater than $AF^2 - AI^2$; alſo $AF^2 - AI^2$ together with EFC, is equal to EIC; therefore, by compoſition, the ratio of IB to BF is greater than that of EIC to EFC; and the ratio of AC×IB to AC×FB greater than that of EIC to EFC: alſo, by permutation, the ratio of AC×IB to EIC greater than the ratio of AC×FB to EFC. But the firſt of theſe ratios is the ſame with that of the ſquare of the radius to the rectangle under the ſine of BAH and the tangent of ABH; and the latter is the ſame with that of the ſquare of the radius to the rectangle under the ſine of BAG and the tangent of ABG; therefore, the latter of theſe two rectangles is greater than the other.

Again, let the triangle BAK be formed, with the angle BAK leſs than BAG, and the perpendicular KL be drawn. Then the rectangle under the ſine of BAK and the tangent of ABK is to the ſquare of the radius, as the ſquare of KL to the rectangle under

AC, BL. Here, FL being to FB as 2 AFL to 2 AFB or EFC, and 2 AFL leſs than $AL^2 - AF^2$, by converſion, the ratio of LB to FB will be greater than the ratio of ELC to EFC; therefore, as before, the rectangle under the ſine of BAG and the tangent of ABG is greater than that under the ſine of BAK and the tangent of ABK.

COROLLARY 1.

BF is equal to the tangent of the circle from the point B; therefore, BF is the tangent, and AB the ſecant, to the radius AC, of the angle, whoſe coſine is to the radius as AC to AB. Therefore, AF is the tangent, to the ſame radius, of half the complement of that angle; and AF is alſo the coſine of the angle BAG to this radius.

COROL. 2.

The ſine of the angle compoſed of the complement of AGB, and twice the complement of ABG, is equal to three times the ſine of the complement of AGB. Let fall the perpendicular AH (Fig. 2.), cutting the circle in I; continue GF to K, and draw AK. Then $BF^2 = EBC = GBL$. Therefore, GB : BF :: BF : BL, and the triangles GBF, FBL are ſimilar. Conſequently FL is perpendicular to GB, and parallel to AH; whence GH being equal to HL, GM is equal to MF, and MK equal to three times GM.

Now, the arc $IK = 2IC + GI$; and the angle $IAK = 2IAC + GAI$; alſo GM is to MK as the

the ſine of the arc GI to the ſine of the arc IK, that is, as the ſine of the angle GAI to the ſine of the angle IAK. Therefore, the ſine of the angle IAK (=2IAC + GAI) is equal to three times the ſine of the angle GAI; but GAI is the complement of AGB, and IAC the complement of ABG.

COROL. 3.

If (Fig. 3.) any line BN be drawn to divide the angle ABG, and AN be joined, alſo AO be drawn perpendicular to BN, and continued to the circle in P, the ſine of the angle compoſed of NAP and 2PAC will be leſs than three times the ſine of the angle NAP. Draw NQR perpendicular to AB, cutting AP in S; join AR, and draw QT perpendicular to BN, and parallel to AO; then $BQ^q = NBT$. But BQ^q is greater than the rectangle EBC, that is, greater than the rectangle NBV, under the two ſegments of the line BN drawn from B, to cut the circle in N and V: therefore, TB is greater than VB, and NO greater than OT. Conſequently NS is greater than SQ. Hence RS is leſs than three times NS; and therefore, the ſine of the angle PAR (=NAP+2PAC) is leſs than three times the ſine of NAP.

PROBLEM I.

To find in the ecliptic the point of longeſt *aſcenſion.*

ANALYSIS.

Let (Fig. 4.) ABC be the equator, ADC the ecliptic, BD the ſituation of the horizon, when D is the point of longeſt aſcenſion. Let EFG be another ſituation of the horizon. Then the ratio of the ſine of EB to the ſine of FD is compounded of the ratio of the ſine of BG to the ſine of GD, and of the ratio of the ſine of AE to the ſine of AF; but the angles B and E being equal, the arcs EG, GB together make a ſemicircle; and, by the approach of EG towards GB, the ultimate magnitude of BG will be a quadrant, and the ultimate ratio of EB to FD will be compounded of the ratio of the radius to the ſine of DG (that is, the coſine of BD) and of the ratio of the ſine of AB to the ſine of AD. Draw the arc DH perpendicular to AB. Then, in the triangle BDH, the radius is to the coſine of BD, as the tangent of the angle BDH to the cotangent of HBD. Alſo, in the triangle BDA, the ſine of AB is to the ſine of AD as the ſine of the angle BDA (or BDC) to the ſine of ABD; therefore, the ultimate ratio of BE to DF is compounded of the ratio of the tangent of BDH to the cotangent of ABD, and of the ratio of the ſine of BDC to the ſine of ABD; which two ratios compound that of the rectangle under the tangent of BDH and the ſine of BDC to the rectangle under the cotangent and the ſine of the given angle ABD.

But, when D is the point of longeſt aſcenſion, the ratio of BE to DF is the greateſt that can be; therefore, then the ratio of the rectangle under the tangent of BDH and the ſine of BDC to the given rectangle under the cotangent and ſine of the given angle ABD muſt be the greateſt that can be; and conſequently, the rectangle under the tangent of BDH, and the ſine of BDC, muſt be the greateſt that can be.

In the triangle BDA, the ſine of BDH is to the ſine of HDA, as the coſine of ABD to the coſine of BAD. Now, in the preceding lemma, let the angle BAG of the triangle AGB be equal to the ſpherical angle BDC: then will the ſum of the angles ABG, AGB be equal to the ſpherical angle BDA. And, if AG in the triangle AGB, be to AB as the coſine of the ſpherical angle DBA to the coſine of DAB, that is, as the ſine of BDH to the ſine of HDA, the angle ABG, in the triangle, will be equal to the ſpherical angle BDH; and the angle AGB, in the triangle, equal to the ſpherical angle HDA. Therefore, by the firſt corollary of the lemma, that the rectangle under the tangent of the ſpherical angle BDH and the ſine of BDC be the greateſt that can be, the coſine of BDC muſt be equal to the tangent of half the complement of the angle, whoſe coſine is to the radius, as AG to AB, in the triangle, or as the coſine of the ſpherical angle ABD to the coſine of the ſpherical angle BAD.

If IK be the ſituation of the horizon, when the ſolſtitial point is aſcending, in the quadrantal triangle AIK, the coſine of KIC is to the radius as the coſine of IKA (= DBA) to the coſine of IAK. Therefore,

fore, the coſine of BDC, when D is the point of longeſt aſcenſion, is equal to the tangent of half the complement of the angle, which the ecliptic makes with the horizon, when the ſolſtitial point is aſcending.

But, the ſine of the angle compoſed of DAB, and twice ABD, muſt be leſs than three times the ſine of the angle BAD. In the ſpherical triangle ABD, the angles BAD, ABD together exceed the external angle BDC. Therefore, in the third corollary of the lemma, let the angle BAN be equal to the ſum of the ſpherical angles BAD, ABD: but here, AN is to AB as the coſine of the ſpherical angle ABD to the coſine of BAD; and AN is alſo to AB as the ſine of ABN to the ſine of ANB, that is, as the coſine of BAP to the coſine of NAP; conſequently, ſince the angle BAN is equal to the ſum of the ſpherical angles BAD, ABD, the angle NAP is equal to the ſpherical angle BAD, and the angle BAP equal to the ſpherical angle ABD; but the ſine of the angle compoſed of NAP and twice PAB is leſs than three times the ſine of NAP; therefore, the ſine of the angle compoſed of the ſpherical angle BAD and 2 ABD will be leſs than three times the ſine of the angle BAD; otherwiſe no ſuch triangle DBA, as is here required, can take place, but the point A will be the point of longeſt aſcenſion.

If the ſine of the angle A be greater than one third of the radius, the point A can never be the point of longeſt aſcenſion; but when the ſine of this angle is leſs, the angle compounded of BAD and twice ABD, may be greater or leſs than a quadrant;

and therefore, the magnitude of the angle ABD, that A be the point of longeſt aſcenſion, is confined within two limits, of which the double of one added to the angle A, as much exceeds a quadrant, as the double of the other added to that angle falls ſhort of it; therefore, double the ſum of thoſe two angles, together with twice A, makes a ſemicircle; and the ſingle ſum of thoſe two angles added to A makes a quadrant.

PROBLEM II.

To find when the arc of the ecliptic differs moſt from its oblique aſcenſion.

ANALYSIS.

If (Fig. 5.) BD be the ſituation of the horizon, when CD differs moſt from CB, as before, the ultimate ratio of BE to DF will be compounded of the ratio of the radius to the ſine of DG (or the coſine of DB) and of the ratio of the ſine of CB to the ſine of CD: but, when CD differs moſt from CB, BE and DF are ultimately equal; therefore, then the coſine of BD is to the radius as the ſine of CB to the ſine of CD.

Draw the arc CHI of a great circle, that DH be equal to DB; then, BH being double BD, half the ſine of BH is to the ſine of BD or DH, as the coſine of BD to the radius; therefore, half the ſine of BH is to the ſine of DH as the ſine of CB to the ſine of CD; but the ſine of the angle BCH is to the ſine of BH as the ſine of the angle CHB to the

ſine

ſine of CB; whence, by equality, half the ſine of BCH is to the ſine of DH as the ſine of CHB to the ſine of CD: but as the ſine of CHB to the ſine of CD, ſo, in the triangle CHD, is the ſine of DCH to the ſine of HD: conſequently, the ſine of DCH is equal to half the ſine of BCH. Hence, the difference of the angles BCH, DCH being given, thoſe angles are given, and the arc CHI is given by poſition.

Moreover, in the triangle BCH, the baſe BH being biſected by the arc CD, the ſine of the angle CHD is to the ſine of the given angle CBD, as the ſine of the given angle HCD to the ſine of the given angle BCD; therefore, the angle CHB is given; in ſo much, that in the triangle CBH all the angles are given.

The ſum of the ſines of the angles BCH, DCH is to the difference of their ſines, as the tangent of half the ſum of thoſe angles to the tangent of half their difference; therefore, the tangent of half the ſum of BCH, DCH is three times the tangent of half BCD.

In (Fig. 6.) the iſoſceles triangle ABC, let the angle BAC be equal to the ſpherical angle BCD, and let AE be perpendicular to BC; alſo, CF being taken equal to CB, join AF: then EF is equal to three times EB; and as EF to EB, ſo is the tangent of the angle EAF to the tangent of EAB; but EAB is equal to half the ſpherical angle BCD: therefore, the angle EAF is equal to half the ſum of the ſpherical angles BCD, BCH; and conſequently, the angle CAF equal to the ſpherical angle DCH. Here, AF is to CF as the ſine of the angle ACF to the ſine of CAF; and CB is to AB as the ſine of the angle BAC to the ſine of ACB: therefore, CF being equal to CB, and the ſine of ACF to the ſine of ACB, by equality, AF is to AB as the ſine of the angle BAC to the ſine of CAF, that is, as the ſine of the ſpherical angle BCD to the ſine of the ſpherical angle DCH.

Let (Fig. 7.) the triangle AGB have the angle ABG equal to the ſpherical angle CBD, and the ſide AG equal to AF. Then, AG is to AB as the ſine of the ſpherical angle BCD to the ſine of the ſpherical angle DCH, that is, as the ſine of the ſpherical angle CBH to the ſine of the ſpherical angle CHB: but AG is to AB alſo as the ſine of the angle ABG to the ſine of AGB; therefore, the angle ABG being equal to the ſpherical angle CBH, the angle AGB is equal to the ſpherical angle CHB: and moreover, when the angle ABG is greater than ABF, that is, when the ſpherical angle CBH is greater than the complement of half BCD, the three angles ABG, AGB and BAC together exceed two right.

Hence, (Fig. 8.) towards the equinoctial point C, where the angle CBD is obtuſe, a ſituation of the horizon, as BD, may always be found, wherein CD more exceeds CB than in any other ſituation: and when the acute angle DBA is greater than the complement of half BCD, another ſituation of the horizon, as KLM, may be found, toward the other equinoctial point A, wherein the arc of the ecliptic CK will be leſs than the arc of the equator, and their difference be greater than in any other ſituation. But, if the angle DBA be not greater than the com-

plement

plement of half BCD, the arc of the ecliptic, between C and the horizon, will never be leſs than the arc of the equator, between the ſame point C and the horizon.

In the two ſituations of the horizon, the angles CHB and KMA are equal.

SCHOLIUM 1.

To find the point in the ecliptic, where the arc of the ecliptic moſt exceeds the right aſcenſion, is a known problem: that point is, where the coſine of the declination is a mean proportional between the radius and the coſine of the greateſt declination.

In the preceding figure, ſuppoſing the angle CBD to be right, then, becauſe when CD moſt exceeds CB, the coſine of BD is to the radius as the ſine of CB to the ſine of CD, and, in the triangle CBD, the ſine of CB is to the ſine of CD as the ſine of the angle CDB to the radius, alſo the ſine of CDB is to the radius as the coſine of BCD to the coſine of BD; therefore, the coſine of BD is to the radius as the coſine of the angle BCD to the coſine of the ſame BD, and the coſine of BD is a mean proportional between the radius and the coſine of BCD.

SCHOLIUM 2.

In any given declination of the Sun, to find when the azimuth moſt exceeds the angle which meaſures the time from noon, is a problem analogous to the preceding.

PROBLEM III.

*The tropic found, by Dr. Halley's method *, without any conſideration of the parabola.*

The obſervations are ſuppoſed to give the proportions between the differences of the ſines of three declinations of the Sun near the tropic; but the ſine of the Sun's place is in a given proportion to the ſine of the declination; therefore, the ſame obſervations give equally the proportion between the differences of the ſines of the Sun's place, in each obſervation.

Now (Fig. 9.), let ACE be the ecliptic, AE its diameter between ♈ and ♎, and its center F; let B, C, D be three places of the Sun; BG, CI, DH the ſines of thoſe places reſpectively. Draw CK, BL parallel to AE, which may meet HD, in N and M. Then, by the obſervations, the ratio of DM to DN is given. Therefore, if BD be drawn to meet KL in O, the ratio of BD to OD is given; and the ratio of BD to DC is alſo given, they being the chords of the given angles BFD, CFD: hence the ratio of CD to DO, in the triangle CDO, is given; and conſequently, the angle COD will be given: which angle is the diſtance of the tropic from the middle point of the ecliptic between B and D: for, FPR being perpendicular to OC, and FQS perpendicular to DB, the angle QFP is equal to QOP, the points O, P, Q, F, being in a circle.

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* Vide Philoſophical Tranſactions, N° 215.

THE CALCULATION.

$$\left.\begin{array}{l} DN : DM \\ \text{ſ. } \tfrac{1}{2}BFD : \text{ſ. } \tfrac{1}{2}CFD \end{array}\right\} :: \text{rad.} : \text{t. } \angle\, \chi$$

$$\text{rad.} : \text{t. } \angle\, \overline{\chi \backsim 45^\circ} :: \text{t. } \tfrac{1}{4}BFC : \text{t. } \frac{COD \backsim DCO}{2}$$

$$\text{If } \chi > 45^\circ,\ \angle\, COD > DCO$$

And

$$\text{if } \chi \angle\, 45^\circ,\ \angle\, COD < DCO.$$

If the intervals between the obſervations are ſo ſmall, that the ſines differ not much from the arches, the arches BC, CD may be counted in time, and the calculation may be abbreviated thus:

$$DM : DN :: \text{arc. } BD : Z \text{ (for } DO)$$

$$DC + Z : 2DC :: \tfrac{1}{4}BC : SR.$$

Or,

$$DM \times DC + DN \times BD : DM \times DC :: \tfrac{1}{2}BC : SR.$$

Fig 1

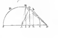

Fig 2

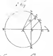

Fig 3

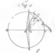

Fig 4

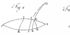

Fig 5

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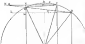

Fig 9